The mod-m diagram monoids

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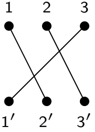
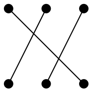
Permutations as diagrams

Let $k \in \mathbb{Z}_{>0}$, $K = \{1, ..., k\}$ and $K' = \{1', ..., k'\}$.
A **permutation** is a bijection from K to K'.

Examples

When $k = 3$,

$(123) =$ 

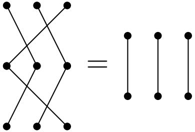
$(132) =$ 

Ie. a permutation can be thought of as lines from one row of vertices to another row of vertices.

Products of permutations

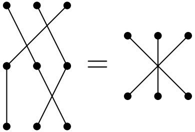
Examples

(123)
(132)



$= \mathrm{id}_3$

(123)
(23)



$= (13)$

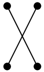
Definition

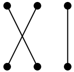
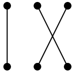
Symmetric group $\mathcal{S}_k$ consists of all permutations endowed with the above product.

Generators of symmetric group

The symmetric group $\mathcal{S}_k$ is generated by **transpositions**.

Examples

$\mathcal{S}_2 = \langle$  $\rangle = \langle \sigma_1 \rangle$

$\mathcal{S}_3 = \langle$  ,  $\rangle = \langle \sigma_1, \sigma_2 \rangle$

$\vdots$

$$\mathcal{S}_k = \langle \sigma_1, ..., \sigma_{k-1} \rangle$$

Presentation of $\mathcal{S}_k$

Proposition

The symmetric group is presented by the $k-1$ transposition generators $\sigma_1, \ldots, \sigma_{k-1}$ along with the relations:

(i) $\sigma_i^2 = \mathrm{id}_k$;

(ii) $\sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j$ for all $|j-i| = 1$;

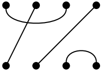
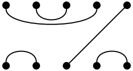
(iii) $\sigma_j \sigma_i = \sigma_i \sigma_j$ for all $|j-i| \geq 2$.

That is, all finite words of transposition generators whose products are equal may be shown as equivalent using the above relations.

Brauer monoid $\mathcal{B}_k$

Brauer monoid $\mathcal{B}_k$ is like the symmetric group $\mathcal{S}_k$ except lines are allowed to connect vertices in the same row.

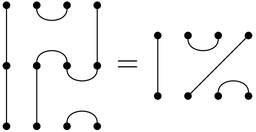
Examples

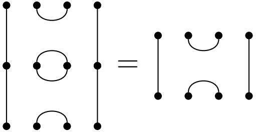
 $\in \mathcal{B}_4$  $\in \mathcal{B}_5$

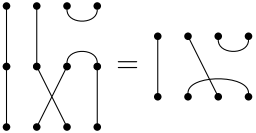
Multiplying elements of $\mathcal{B}_k$

Similar to multiplying permutations, except loops may form in the middle which are *forgotten*.

Examples



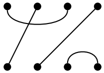


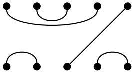


Jones monoid $\mathcal{J}_k$

Jones monoid $\mathcal{J}_k$ (aka the Temperley-Lieb monoid) consists of all elements of $\mathcal{B}_k$ that may be drawn without any lines crossing (all within convex hull of vertices).

Examples

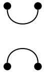
 $\notin \mathcal{J}_4$

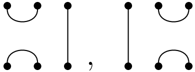
 $\in \mathcal{J}_5$

Generators of the Jones monoid

The Jones monoid $\mathcal{J}_k$ is generated by **diapsis generators**.

Examples

$\mathcal{J}_2 = \Big\langle$  $\Big\rangle = \langle \mathfrak{d}_1 \rangle$

$\mathcal{J}_3 = \Big\langle$  $\Big\rangle = \langle \mathfrak{d}_1, \mathfrak{d}_2 \rangle$

$$\vdots$$

$$\mathcal{J}_k = \langle \mathfrak{d}_1, \dots, \mathfrak{d}_{k-1} \rangle$$

Note: Throw in transposition generators to generate the Brauer monoid $\mathcal{B}_k$.

Presentation of $\mathcal{J}_k$

Proposition

The Jones monoid $\mathcal{J}_k$ is presented by the $k-1$ diapsis generators $\mathfrak{d}_1, \ldots, \mathfrak{d}_{k-1}$ along with the relations:

(i) $\mathfrak{d}_i^2 = \mathfrak{d}_i$;

(ii) $\mathfrak{d}_i\mathfrak{d}_j\mathfrak{d}_i = \mathfrak{d}_i$ for all $|j-i| = 1$;

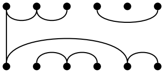
(iii) $\mathfrak{d}_j\mathfrak{d}_i = \mathfrak{d}_i\mathfrak{d}_j$ for all $|j-i| \geq 2$.

(Planar) partition monoid

(i) **Partition monoid** $\mathcal{P}_k$ allows **generalised lines**, which are often called **blocks** (call elements of $\mathcal{P}_k$ **bipartitions**).

(ii) **Planar partition monoid** $\mathbb{P}\mathcal{P}_k$ is the submonoid of bipartitions which may be drawn without blocks crossing.

Example

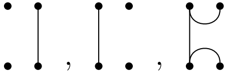


$= \{\{1,2,3,1',5',6'\},\{4,6\},\{5\},\{2',3',4'\}\} \in \mathbb{P}\mathcal{P}_6$

Generators of planar partition monoid $\mathbb{P}\mathcal{P}_k$

The planar partition monoid $\mathbb{P}\mathcal{P}_k$ is generated by **monapsis generators** and $(2,2)$-**transapsis generators** (Halverson & Ram).

Examples

$\mathbb{P}\mathcal{P}_2 = \Big\langle$  $\Big\rangle = \langle \mathfrak{a}_1^1, \mathfrak{a}_2^1, \mathfrak{t}_1 \rangle$

$\vdots$

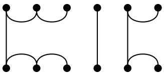
$$\mathbb{P}\mathcal{P}_k = \langle \mathfrak{a}_1^1, \ldots, \mathfrak{a}_k^1, \mathfrak{t}_1, \ldots, \mathfrak{t}_{k-1} \rangle$$

(Planar) uniform block bijections

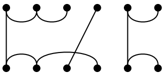
(i) **the monoid of uniform block bijections** $\mathfrak{F}_k$ is the set of all bipartitions $\alpha \in \mathcal{P}_k$ such that for all blocks $b \in \alpha$, $|U(b)| = |L(b)|$;

(ii) **the monoid of planar uniform block bijections** $\mathbb{P}\mathfrak{F}_k$ is the set of all planar bipartitions $\alpha \in \mathbb{P}\mathcal{P}_k$ such that for all blocks $b \in \alpha$, $|U(b)| = |L(b)|$.

Example



$\in \mathbb{P}\mathfrak{F}_6$



$\notin \mathbb{P}\mathfrak{F}_6$

$\in \mathfrak{F}_6$

Generators for monoid of (planar) uniform block bijections

(i) monoid of planar uniform block bijections $\mathbb{P}\mathfrak{F}_k$ is generated by $(2,2)$-transapsis generators; and

(ii) monoid of uniform block bijections $\mathfrak{F}_k$ is generated by transposition generators and $(2,2)$-transapsis generators.

Presentation of $\mathfrak{F}_k$

Proposition

(Kosuda, East/FitzGerald) The monoid of uniform block bijections $\mathbb{P}\mathfrak{F}_k$ is presented by the transposition generators σ_1, ..., σ_{k-1} and the $(2,2)$-transapsis generators $\mathfrak{t}_1, \ldots, \mathfrak{t}_{k-1}$ along with the relations:

(i) (I) $\sigma_i^2 = \mathrm{id}_k$;
(II) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ for all $|j-i| = 1$;
(III) $\sigma_j\sigma_i = \sigma_i\sigma_j$ for all $|j-i| \geq 2$.

(ii) (I) $\mathfrak{t}_i^2 = \mathfrak{t}_i$;
(II) $\mathfrak{t}_j\mathfrak{t}_i = \mathfrak{t}_i\mathfrak{t}_j$ for all $|j-i| \geq 1$;

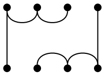
(iii) (I) $\sigma_i\mathfrak{t}_i = \mathfrak{t}_i = \sigma_i\mathfrak{t}_i$;
(II) either $\sigma_j\mathfrak{t}_i\sigma_j = \sigma_i\mathfrak{t}_j\sigma_i$ or $\sigma_j\sigma_i\mathfrak{t}_j = \mathfrak{t}_i\sigma_j\sigma_i$ for all $|j-i| = 1$;
(III) $\sigma_j\mathfrak{t}_i = \sigma_j\mathfrak{t}_i$ for all $|j-i| \geq 2$.

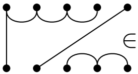
(Planar) mod-m monoid

(i) **mod-m monoid** $\mathfrak{M}_k^m$ is the set of all bipartitions $\alpha \in \mathcal{P}_k$ such that for all blocks $b \in \alpha$, $|U(b)| \equiv |L(b)| \pmod{m}$;

(ii) **Planar mod-m monoid** $\mathbb{P}\mathfrak{M}_k^m$ is the set of all planar bipartitions $\alpha \in \mathbb{P}\mathcal{P}_k$ such that for all blocks $b \in \alpha$, $|U(b)| \equiv |L(b)| \pmod{m}$.

Example

 $\in \mathbb{P}\mathfrak{M}_4^2$

 $\in \mathbb{P}\mathfrak{M}_5^3$

Note:

(i) $\mathfrak{M}_k^1 = \mathcal{P}_k$;

(ii) $\mathbb{P}\mathfrak{M}_k^1 = \mathbb{P}\mathcal{P}_k$;

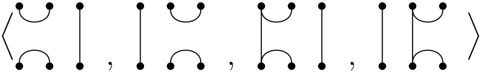
(iii) $\mathfrak{M}_k^m = \mathfrak{F}_k$ for all $k < m$;

(iv) $\mathbb{P}\mathfrak{M}_k^m = \mathbb{P}\mathfrak{F}_k$ for all $k < m$.

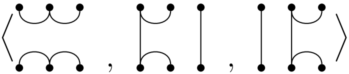
Generators of planar mod-m monoid

The planar mod-m monoid $\mathbb{P}\mathfrak{M}_k^m$ is generated by m-**apsis generators** and $(2, 2)$-transapsis generators.

Examples

$\mathbb{P}\mathfrak{M}_3^2 =$  $= \langle \mathfrak{a}_1^2, \mathfrak{a}_2^2, \mathfrak{t}_1, \mathfrak{t}_2 \rangle$

$\vdots$

$\mathbb{P}\mathfrak{M}_3^3 =$  $= \langle \mathfrak{a}_1^3, \mathfrak{t}_1, \mathfrak{t}_2 \rangle$

$\vdots$

$$\mathbb{P}\mathfrak{M}_k^m = \langle \mathfrak{a}_1^m, \ldots, \mathfrak{a}_{k-m+1}^m, \mathfrak{t}_1, \ldots, \mathfrak{t}_{k-1} \rangle$$

Planar mod-2 monoid

Cardinality of $\mathbb{P}\mathfrak{M}_k^2$ is $\frac{\binom{3k}{k}}{2k+1}$ (A001764 on OEIS), which is the same as:

(i) number of non-crossing partitions of $[2k]$ with all blocks of even size;

(ii) number of ternary trees with k internal nodes;

(iii) Pfaff-Fuss-Catalan sequence for $m = 3$ (cardinality of Jones monoid is the Catalan numbers, which is PFC sequence for $m = 2$);

(iv) number of lattice paths of k east steps and $2k$ north steps from $(0, 0)$ to $(k, 2k)$ and lying weakly below the line $y = 2x$;

(v) number of lattice paths from $(0, 0)$ to $(2n, 0)$ that do not cross below the x-axis using step-set $\{(1, 1), (0, -2)\}$.

Presentation of $\mathbb{P}\mathfrak{M}_k^2$

Conjecture

The planar mod-2 monoid $\mathbb{P}\mathfrak{M}_k^2$ is presented by the generators:

(i) id_k (identity);

(ii) $\mathfrak{d}_1, \ldots, \mathfrak{d}_{k-1}$ (diapsis generators);

(iii) $\mathfrak{t}_1, \ldots, \mathfrak{t}_{k-1}$ ($(2,2)$-transapsis generators),

along with the relations:

(i)
- (I) $\mathfrak{d}_i^2 = \mathfrak{d}_i$;
- (II) $\mathfrak{d}_i\mathfrak{d}_j\mathfrak{d}_i = \mathfrak{d}_i$ for all $|j - i| = 1$;
- (III) $\mathfrak{d}_j\mathfrak{d}_i = \mathfrak{d}_i\mathfrak{d}_j$ for all $|j - i| \geq 2$;

(ii)
- (I) $\mathfrak{t}_i^2 = \mathfrak{t}_i$;
- (II) $\mathfrak{t}_j\mathfrak{t}_i = \mathfrak{t}_i\mathfrak{t}_j$ for all $|j - i| \geq 1$;

(iii)
- (I) $\mathfrak{d}_i\mathfrak{t}_i = \mathfrak{d}_i = \mathfrak{t}_i\mathfrak{d}_i$;
- (II) $\mathfrak{t}_i\mathfrak{d}_j\mathfrak{t}_i = \mathfrak{t}_i\mathfrak{t}_j$ for all $j - i = 1$; and
- (III) $\mathfrak{t}_j\mathfrak{d}_i = \mathfrak{d}_i\mathfrak{t}_j$ for all $|j - i| \geq 2$.

progress

(i) have verified up to $k = 7$ using GAP, would be pretty odd for it to change after that given the generators all commute for $|j - i| \geq 2$;

(ii) I'm reasonably confident I have a bound on normal form words, however I've been unable to get the bound down to the cardinality in order to reach the desired conclusion (can explain more to anyone interested!).

Other results

(i) characterisation of $\mathfrak{A}_k^m$, $\mathbb{X}\mathfrak{A}_k^m$;

(ii) recurrence relations for cardinality of $\mathbb{P}\mathfrak{M}_k^m$, $\mathfrak{M}_k^m$, $\mathfrak{A}_k^m$, $\mathbb{X}\mathfrak{A}_k^m$;

(iii) characterisation of Green's $\mathcal{D}$ relation for all submonoids of $\mathcal{P}_k$ that are closed under vertical flips;

(iv) recurrence relations for number of Green's $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$ relations for $\mathbb{P}\mathfrak{M}_k^m$, $\mathfrak{M}_k^m$.

Thanks for listening! Questions?